# THE THEORY OF PLASTICITY WHICH TAKES INTO ACCOUNT RESIDUAL MICROSTRESSES

## (TEORIIA PLASTICHNOSTI UCHITYVAIUSHCHAIA OSTATOCHNYE MIKRONAPRIAZHENIIA)

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As a condition of "active" deformation usually the following inequality is accepted

$$dT > 0 \tag{0.1}$$

where

$$T = \sqrt{\frac{1}{2}\sigma'_{ij}\sigma'_{ij}} =$$

$$= \frac{1}{\sqrt{6}}\sqrt{(\sigma_{xx}-\sigma_{yy})^2+(\sigma_{yy}-\sigma_{zz})^2+(\sigma_{zz}-\sigma_{xx})^2+6(\sigma_{xy}^{\ 2}+\sigma_{xz}^{\ 2}+\sigma_{yz}^{\ 2})}$$

$$\sigma_{ij}' = \sigma_{ij} - \frac{1}{3}\sigma\delta_{ij}, \qquad \sigma = \sigma_{ii} \tag{0.2}$$

Thus, the boundary which separates the region of elastic deformations from the region of plastic deformations, is determined by the equation

$$T = C \tag{0.3}$$

where $C$ is the intensity of the shear stresses $T$ at the instant of loading under consideration. According to (0.3) in the process of active deformation the yield surface gradually expands in all directions, remaining similar to its initial shape. However, in reality, the deformation not only alters the dimensions of the yield surface but also changes its shape. Moreover, it causes the displacement of the yield surface as a whole, and as a result of this, the point $\sigma_{ij} = 0$ ceases to be the center of the region of elastic deformations.

Recently much attention was devoted to explaining the changes of shape of the yield surface [1,2,3]. Very little attention was devoted, from the theoretical point of view, to the problem of the displacement of the yield surface as a whole. The presence, however, of a very pronounced Bauschinger effect in most materials indicates the fact that

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such displacements, without any doubt, take place and that they are quite considerable. (Parenthetically, the usual equation of the flow boundary (0.3) gives a Bauschinger effect with a reversed sign, i.e. leads to its erroneous evaluation not only quantitatively but qualitatively as well). The present paper contains an attempt to develop a theory of plasticity which would incorporate the displacements of the center of the region of elastic deformations. We shall, however, neglect the changes of the shape of the yield surface, assuming that the role of this factor can be sufficiently appraised on the basis of the results obtained by other authors.

The investigations of F. Edelman and D.C. Drucker [4] and of A.Iu. Ishlinskii [11] treat the same problem. In [4] means of constructing the theory of plasticity which takes into account the Bauschinger effect is only slightly indicated. In [11] another version of the theory of plasticity is proposed which is based on the assumptions that the strain hardening is linear, and that the yield surface is displaced as a rigid body. This version of the theory, which deserves serious attention on its own merits, is included in the following theory as one of its limiting cases, (the other one is the classical theory of plastic flow).

## 1. Relations between stresses and plastic strains

Let the yield surface at the instant the first plastic deformations occur be determined by the equation

$$T = C_T \tag{1.1}$$

where

$$C_T = \frac{1}{\sqrt{3}} \sigma_T \tag{1.2}$$

and $\sigma_T$ is the yield stress in simple tension. If we assume that in the process of deformation the yield surface remains similar to its initial form, but that it undergoes at the same time some translatory displacement, then its equation (at an arbitrary instant of loading) will be expressed as

$$T^\circ = C^\circ \tag{1.3}$$

where

$$T^\circ = \sqrt{\tfrac{1}{2} \sigma_{ij}^{\circ\prime} \sigma_{ij}^{\circ\prime}}, \qquad \sigma_{ij}^\circ = \sigma_{ij} - s_{ij}, \qquad \sigma_{ij}^{\circ\prime} = \sigma_{ij}^\circ - \tfrac{1}{3} \sigma^\circ \delta_{ij}, \quad \sigma^\circ = \sigma_{ii}^\circ \tag{1.4}$$

and where a constant $C^\circ$ corresponds to the value of the invariant $T^\circ$ at the instant of loading considered. The components of the symmetric tensor of the second rank $s_{ij}$, introduced above, (it has dimensions of the stress tensor), are the coordinates of the center of the region of elastic deformations in the system $\sigma_{ij}$.

This tensor has the following obvious properties:

(a) it is equal to zero at the instant of appearance of first plastic deformations, since then the equality (1.3) has to reduce to (1.1);

(b) it remains constant at the neutral loading, (that is at a loading along the yield surface), since in this case the region of elastic deformations must remain at rest;

(c) $s_{ij}$ vary during the active loading, and are determined by the plastic strains.

We shall call $s_{ij}$ the residual microstress tensor, and $\sigma_{ij}{}^0$ the active stress tensor. The meaning of this terminology will be clear from what follows.

Let us assume that the plastic strain increment tensor is completely determined by the active stress tensor and its increment. Then, by analogy with the most accepted version of the theory of plastic flow, we can write, (for materials which exhibit isotropic behavior under simple loads), the following relations

$$d\varepsilon_{ij}^p = \sigma_{ij}{}^{\circ\prime}\, df(T^\circ) \tag{1.5}$$

Here $f(T^\circ)$ is the loading function, to be determined experimentally. Its mechanical significance follows from the equality

$$dA^\circ = \sigma_{ij}{}^{\circ\prime} d\varepsilon_{ij}{}^p = 2T^{\circ 2} df \tag{1.6}$$

From this

$$A^\circ = \int_{T_0}^{T^\circ} 2T^{\circ 2}\frac{df}{dT^\circ}\, dT^\circ \tag{1.7}$$

Thus, $f(T^\circ)$ is directly related to the work of the active stresses along the plastic strain paths. Furthermore, as it follows from (1.7), in the relations (1.5) a hypothesis is made to the effect that the above-mentioned work depends only on the initial and terminal values of the intensity of the active shear stresses.

The case when the yield surface is determined by the equation

$$T^\circ = C_T = 1/\sqrt{3}\,\sigma_T = \text{const} \tag{1.8}$$

should be considered separately. In this case it preserves, in the process of the deformation, not only its form, but also all its dimensions, thus being displaced in the $\sigma_{ij}$ space as a rigid body. Here the procedure should be similar to that in Reuss' theory, i.e. (1.5) is replaced by a relation

$$d\varepsilon_{ij}{}^p = \sigma_{ij}{}^{\circ\prime} d\lambda \tag{1.9}$$

The material which obeys (1.8) will be called the material with "an ideal Bauschinger effect". For such a material this effect is precisely equivalent to the effect of strainhardening taken with reversed sign. Usually, however, the Bauschinger effect is less pronounced than the strainhardening effect. The relationships (1.5) or (1.8), (1.9) are in themselves not sufficient to determine the plastic strain path from a given loading path (or vice versa), because at the loading the true stress tensor $\sigma_{ij}$ is given and not the tensor $\sigma_{ij}^0$. Thus, the above given formulas should be supplemented by the relationships between the tensor $s_{ij}$ and plastic strains. However, in connection with this, an immediate question arises: what should be the form of this relationship? Should it have a form of non-integrable differential relations analogous to (1.5), or should it be thought of in the form of a functional dependence, which directly expresses $s_{ij}$ by $\epsilon_{ij}^p$? In the next section it will be demonstrated, on the basis of physical considerations, that the latter appears to be more probable. Based upon this consideration, we shall write

$$\varepsilon_{ij}^p = \frac{1}{2g} s_{ij} \tag{1.10}$$

where $g$ is a function of the invariants of the tensor $s_{ij}$.

In the sequel we shall assume that $g$ is a function of $T_s$ only, where

$$T_s = \sqrt{\tfrac{1}{2} s_{ij} s_{ij}} \tag{1.11}$$

The theory of plasticity based on the formulas (1.8), (1.9) and (1.10) (for $g = g_0$ = const.) was advanced by A.Iu. Ishlinskii, [ 11 ].

It remains to determine the elastic strains. We shall assume, (as is always done in the theory of plasticity), that the tensor of elastic strains is connected with the stress tensor $\sigma_{ij}$ by Hooke's law

$$\sigma_{ij} = K\varepsilon_{ii}^y \delta_{ij} + 2G(\varepsilon_{ij}^y)' \tag{1.12}$$

where $K$ is the bulk modulus, $G$ the shear modulus, $(\epsilon_{ij}^y)'$ are the components of the elastic strain deviator tensor. In conclusion of this Section, we would like to mention that according to (1.5) or (1.9) the plastic strain tensor is identical with its deviator. Thus, according to (1.10), the residual stress tensor $s_{ij}$ has the same property. It is possible to formulate a more complicated variant of this theory, which will not neglect the plastic changes of volume, and accordingly, will not neglect the residual mean normal stress. However, it would hardly be appropriate to dwell on this any longer in this paper, whose aim it is to present the basic ideas of the proposed theory and not to exhaust all the possibilities inherent in it.

## 2. Some physical considerations which support the proposed formulas

The relationships (1.5) were written down by analogy with the theory of flow. The relationships (1.10), based on the notion that the tensors $s_{ij}$ and $\epsilon_{ij}^{p}$ depend upon one another according to the principle of elastic interrelation, were proposed without any supporting argument. However, neither the relationship (1.5) nor (1.10) is at all obvious. As a matter of fact, (1.5) asserts, for instance, that the plastic strain increment tensor is similar to the active stress tensor, $\sigma_{ij}^{0}$. The question is asked, why is it similar to this tensor and not to the true stress tensor $\sigma_{ij}$ ? It is also not clear why $s_{ij}$ and $\epsilon_{ij}^{p}$ should be connected by the relationships which are independent of the deformation paths. To clarify all this we shall introduce the following considerations.

It was noticed already long ago that there exists a certain analogy between the resistance to plastic deformation and dry friction. Thus, for instance, the behavior of an ideally elasto-plastic body in simple tension and compression could be represented by the displacement of one end of a spring, whose other end is attached to the body placed on a horizontal plane, (Fig. 1).

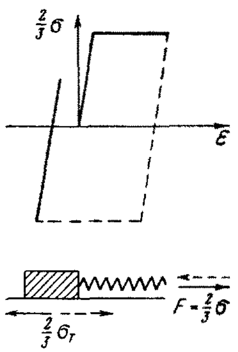


Fig. 1.

This analogy could be extended to the case of combined application of two stresses, say of a normal stress $\sigma_{xx} = \sigma$ and shear stress $\sigma_{xy} = \tau$. In this case the equation of the yield surface is

$$F = \sqrt{X^2 + Y^2} = \tfrac{2}{3}\sigma_T \tag{2.1}$$

where

$$X = \tfrac{2}{3}\sigma, \quad Y = \frac{2}{\sqrt{3}}\tau \tag{2.2}$$

The components of the strain deviator tensor $\epsilon'_{xx}$ and $\epsilon'_{xy}$ corresponding to stresses $\sigma$ and $\tau$ may be compared with the displacements $u$ and $v$ of the ends of two springs, attached at a right angle to a body placed on a horizontal plane. If the forces $X$ and $Y$ satisfy the equality (2.1), then the frictional force is balanced by the resultant of the forces in both springs.

Next, let one of the forces ($X$, say) be increased by an infinitesimal amount $\Delta X$ ( $\Delta F$ is considered to be always positive, $\Delta F > 0$). Then the body starts to move on the plane in the direction of the resultant of the $X$ and $Y$ forces and not in the direction of the increased force. This is because in the former direction the frictional force is balanced.

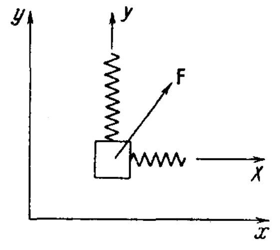


Fig. 2.

Thus, from the mechanical analogy presented above, it follows that the plastic strain increment tensor should be coaxial with the stress tensor, as it is commonly accepted in the theory of plastic flow, and not with the stress increment tensor.

There are three possible ways to extend this analogy to materials which exhibit a strain-hardening effect:

(a) to consider that strain hardening is an irreversible effect and can be interpreted as a continuous increase of the frictional force in the process of active deformation;
(b) to consider that strain hardening is caused by the internal elastic forces which are resisting plastic deformation;
(c) a combination of (a) and (b).

If we accept (a), then the above shown mechanical model (based on considerations used previously) would lead us to the concept of the yield surface expanding uniformly in all directions, and to the coaxiality of the tensors $d\epsilon_{ij}^{p}$ and $\sigma_{ij}^{p}$, i.e. would lead us to the hypotheses of the plastic flow theory.

If we assume that the effect of strain hardening is due to the elastic forces then we get another picture shown in Fig.3.

Consider, as before, a body which is placed on a plane and is under the action of two forces $X$ and $Y$ by means of two mutually perpendicular springs. In this case we have to attach two more springs to the body opposite to the previous ones (Fig.4). The conditions for balancing the frictional force (equation of the yield surface) in this case will be expressed in the following way:

$$\sqrt{(X-X_1)^2+(Y-Y_1)^2}=\tfrac{2}{3}\sigma_T \tag{2.3}$$

where $X_1$ and $Y_1$ are the forces in the additional springs.

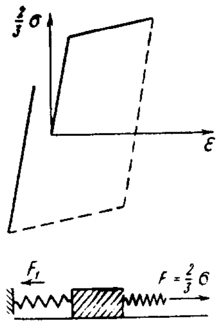

Fig. 3.

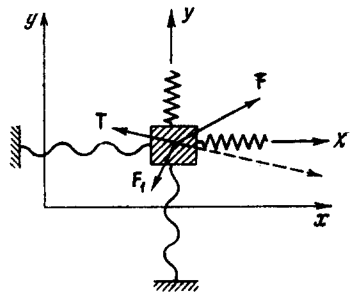

Fig. 4.

Thus, in the case under consideration, the yield surface (in the plane $x = 2/3\,\sigma$, $y = 2/\sqrt{3}\,\tau$) is represented by a circle with constant radius $2/3\,\sigma_T$ and with the center at

$$X_1 = s_{xx} = s, \quad Y_1 = \frac{2}{\sqrt{3}} s_{xy} = \frac{2}{\sqrt{3}} t$$

If the force $X$ is increased by an infinitesimal amount $\Delta X$, (such that the resultant of the forces $X$, $Y$, $X_1$ and $Y_1$ would exceed the frictional force), then the body starts to move to a new equilibrium position. The vector of this displacement will be directed along the resultant of the forces $F(x,y)$ and $F_1(x,y)$ (shown in Fig.4 by a dotted line), and not along the direction of the external force $F(x,y)$ nor along its increment. This is so because the frictional force is balanced only in the direction of the resultant of the forces $F(x,y)$ and $F_1(x,y)$.

It is easy to notice that equations (1.8), (1.9) and (1.10) in the theory of plasticity correspond to the behavior of the above-described mechanical system.

In fact, the equality (1.8) asserts that the yield surface does not change during the deformation — neither its shape nor its size. The relationship (1.9) asserts that the tensors $d\epsilon_{ij}^p$ and $\sigma_{ij}' - s_{ij}$ are similar and therefore coaxial. Finally, formulas (1.10) assert that $\epsilon_{ij}^p$ is an elastic strain tensor with respect to the stresses $s_{ij}$. The physical significance of $s_{ij}$ consists of the following: they are these hidden "internal" ([ 5 ], pp. 136-137) elastic microstresses which appear in the body during the plastic deformation. After the removal of the loading these stresses remain in it since they cannot by themselves overcome frictional forces which oppose plastic deformation.

It is also possible to analyze the behavior of a mechanical system represented by (c), where strain hardening is not totally an elastic effect. In this case the condition for balancing the frictional force in the two-dimensional case will have the following form

$$\sqrt{(X - X_1)^2 + (Y - Y_1)^2} = \rho \tag{2.4}$$

where $X$, $Y$, $X_1$ and $Y_1$ are related to the stresses $\sigma$, $\tau$, $s$ and $t$ as indicated above. In (2.4) $\rho$ represents an invariant quantity which is monotonically increasing during the active deformation, and it is bounded by the following inequality:

$$\frac{2}{3}\sqrt{\sigma^2+3\tau^2}=\sqrt{X^2+Y^2}\geqslant\rho\geqslant\frac{2}{3}\sigma_T \qquad (2.5)$$

For the lower bound of $\rho$ we will get a yield surface (2.3), which corresponds to an ideal Bauschinger effect, and for the upper bound a uniformly expanding yield surface with stationary center given by (0.3).

Applying to the case (c) the same considerations as previously, we come to the conclusion that in this case the plastic strain increment tensor should be coaxial with the active stress tensor $\sigma_{ij}^0 = \sigma_{ij} - s_{ij}$, and not with the true stress tensor $\sigma_{ij}$. Moreover, $s_{ij}$, as in case (b), represent elastic residual stresses which characterize the displacement of the center of the yield surface.

Formulas (1.5) and (1.10) correspond to the case (c). Furthermore, the function $f(T^0)$ represents in these formulas the plastic part of the strain hardening, while the function $g(T_s)$ represents its elastic part.

These two functions have to be determined experimentally from tension tests, followed by compression.

The plastic characteristics of real quasi-isotropic bodies are best described by the scheme (c).

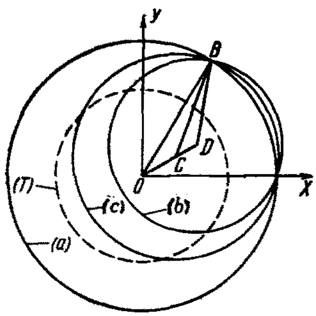


Fig. 5.

It should be mentioned that the material characterized by the scheme (b), i.e. material exhibiting an ideal Bauschinger effect is, because of its mechanical properties, closer to real materials than that which is characterized by the scheme (a). This is obvious from Fig.5, where $OB=F$, $OD = F_1$, $OC = F_2$. The circle ($a$) represents the yield surface with fixed center; circle ($b$) represents the yield surface displaced as a rigid body; circle ($c$) represents the yield surface which exhibits the Bauschinger effect more or less in the same way as it is encountered in the real materials; the circle $T$ is an initial yield surface. In view of this, the case of an ideal Bauschinger effect which is described by formulas (1.8), (1.9) and (1.10) represents considerable interest and deserves further study.

From the foregoing it is now clear why in the proposed theory the plastic strain increment tensor was assumed to be of the same type as the active stress tensor $\sigma_{ij}^0 = \sigma_{ij} - s_{ij}$, and not as the tensor $\sigma_{ij}$. It is also clear why the components $s_{ij}$ were expressed in terms of the plastic strains in a similar way to the principle of elastic relationships. Furthermore, the terminology used to denote $s_{ij}$ as "residual" stresses is now clear, for the stresses $s_{ij}$ remain in the body even after the removal of the external loads, since they cannot themselves overcome plastic resistance.

*Remark 1.* In paper [6] attention was directed to a contradiction in which one may be trapped in drawing an analogy between friction and resistance to plastic deformations, if an attempt is made to account for the influence of the mean normal stress on plastic shear. The modern theory of plasticity, however, generally neglects this influence, and we followed, like others, this pattern. There is no difficulty, however, if it should be required in introducing into a proposed theory the corresponding refinement by including in the theory the well-known notion of a plastic potential.

Remark 2. In drawing above an analogy with dry friction, we limited ourselves to the analysis of a particular case, where only two stresses, $\sigma_{xx} = \sigma$ and $\sigma_{xy} = \tau$, were different from zero. However, the same analogy can be extended to the most general case, namely, to any case of plastic resistance of the quasi-isotropic materials and for an arbitrary loading. Indeed, it can be represented by friction acting on a body being displaced in five-dimensional space [7]. In this case to each deviator there corresponds some vector, and the initial yield surface is a sphere with the center at the origin of the coordinate system. In a two-dimensional case the aforementioned space degenerates into a Euclidian plane and an abstract picture of a hyper-body which is displaced in a five-dimensional space is reduced to the simpler picture which we used before.

## 3. Integration of relationships between the stresses and strains for some special forms of combined loading

In the experimental work quite frequently the following loading paths are used.

(A) Thin-walled circular cylinder with initial stresses $\sigma_{11} = \sigma_0$, $\sigma_{12} = \tau_0$ and initial residual stresses $s_{11} = s_0$, $s_{12} = t_0$ is subjected to additional extension up to $\sigma_{11} = \sigma$ with a constant $\sigma_{12} = \tau = \tau_0$.

(B) The same case, but instead of being subjected to the additional tension, the cylinder is subjected to an additional twist up to the stresses $\sigma_{12} = \tau$ (with $\sigma_{11} = \sigma_0$).

(C) Thin-walled circular cylinder, initially prestressed into the plastic range up to the stress $\sigma = \sigma_0$, and then subjected simultaneously to tension and torsion, the stresses being varied in accordance with the law

$$\sigma - \sigma_0 = c\sqrt{3}\,\tau \qquad (c = \text{const}) \tag{3.1}$$

We shall now determine plastic strains for these three cases assuming an ideal Bauschinger effect and a linear strain hardening. In all the above formulated problems

$$\begin{gathered}\sigma_{11} = \sigma_1, \qquad \sigma_{22} = \sigma_{33} = 0, \qquad \sigma_{12} = \tau, \qquad \sigma_{13} = \sigma_{23} = 0 \\ s_{11} = s, \qquad s_{22} = s_{33} = \tfrac{1}{2} s, \qquad s_{12} = t, \qquad s_{23} = s_{13} = 0\end{gathered} \tag{3.2}$$

In accordance with this, (1.8) has the form:

$$\left(\tfrac{2}{3}\sigma - s\right)^2 + \tfrac{4}{3}(\tau - t)^2 = \tfrac{4}{9}\sigma_T^2 \tag{3.3}$$

and (1.9) and (1.10) are reduced to

$$d\varepsilon_{11}^p = \left(\tfrac{2}{3}\sigma - s\right)d\lambda, \qquad d\varepsilon_{12}^p = (\tau - t)\,d\lambda \tag{3.4}$$

$$\varepsilon_{11}^p = \frac{1}{2g_0}s, \qquad \varepsilon_{12}^p = \frac{1}{2g_0}t \tag{3.5}$$

Because of the assumption of linearity of the strain hardening, $g_0$ is constant. Eliminating $\epsilon_{ij}^p$ from (3.4) and (3.5) we arrive at the following expressions which connect true stresses with residual stresses

$$ds = \left(\tfrac{2}{3}\sigma - s\right)d\lambda^*, \qquad dt = (\tau - t)\,d\lambda^*, \qquad \lambda^* = 2g_0\lambda \tag{3.6}$$

(A) Let $\sigma = \sigma_0$, $\tau = \tau_0$, $s = s_0$, $t = t_0$ for $\lambda^* = 0$, and let $\sigma$ be increased and $\tau$ kept constant. From (3.6) it follows then

$$t = Ce^{-\lambda^*} + \tau_0 = (t_0 - \tau_0)\,e^{-\lambda^*} + \tau \tag{3.7}$$

Because of (3.7) and (3.3) we obtain further

$$\tfrac{2}{3}\sigma - s = \tfrac{2}{3}\sigma_T\sqrt{1 - k^2 e^{-2\lambda^*}}, \qquad k = \frac{\sqrt{3}}{\sigma_T}(\tau_0 - t_0) \tag{3.8}$$

Substituting (3.8) into the right-hand side of the first equality in (3.6) and integrating, taking into account the initial conditions and relationships which must exist between them because of (3.3), we have

$$s = s_0 - \tfrac{2}{3}\sigma_T(x - x_0) + \tfrac{1}{3}\sigma_T \ln\left(\frac{1+x}{1+x_0}\,\frac{1-x_0}{1-x}\right) \tag{3.9}$$

where

$$x = \sqrt{1 - k^2 e^{-2\lambda^*}}, \qquad x_0 = \frac{\sigma_0 - 3/2\,s_0}{\sigma_T} \tag{3.10}$$

Formulas (3.7) and (3.9) may be reduced to

$$s = \frac{2}{3}[\sigma - \sigma_T \operatorname{th} u], \quad t = \tau_0 - \frac{\sigma_T}{\sqrt{3}} \frac{1}{\operatorname{ch} u} \tag{3.11}$$

Here

$$u = \frac{\sigma - \sigma_0}{\sigma_T} + \alpha_0, \quad \operatorname{th} \alpha_0 = x_0 \tag{3.12}$$

(B) This case differs from the previous one only in that during the process of loading $\tau$ varies and $\sigma$ remains constant, and $\sigma = \sigma_0$. Relations (3.6) can be integrated in an analogous way to the case (A). The following are final formulas

$$S = \frac{2}{3}\left(\sigma_0 - \frac{\sigma_T}{\operatorname{ch} V}\right), \quad t = \tau - \frac{\sigma_T}{\sqrt{3}} \operatorname{th} V \tag{3.13}$$

where

$$V = \frac{\sqrt{3}}{\sigma_T}(\tau - \tau_0) + \beta_0, \qquad \operatorname{th} \beta_0 = \frac{\sqrt{3}}{\sigma_T}(\tau_0 - t_0) \tag{3.14}$$

(C) To study this case, define a new variable $\phi$ by the following equations

$$\frac{2}{3}\sigma - s = \frac{2}{3}\sigma_T \cos\varphi, \quad \tau - t = \frac{\sigma_T}{\sqrt{3}} \sin\varphi \tag{3.15}$$

Thus, (3.3) is identically satisfied. Determining $s$ and $t$ from (3.15) and substituting into (3.6) we get

$$d\sigma + \sigma_T \sin\varphi \, d\varphi = \sigma_T \cos\varphi \, d\lambda^* \tag{3.16}$$

$$d\tau - \frac{\sigma_T}{\sqrt{3}} \cos\varphi \, d\varphi = \frac{\sigma_T}{\sqrt{3}} \sin\varphi \, d\lambda^* \tag{3.17}$$

From this

$$\sqrt{3} \cos\varphi \, d\tau - \sin\varphi \, d\sigma = \sigma_T d\varphi \tag{3.18}$$

In the case (C) there exists a relationship between $\sigma$ and $\tau$ during the process of loading, namely, relationship (3.1), which is

$$d\sigma = \sqrt{3}\, c\, d\tau \tag{3.19}$$

Substituting (3.19) in (3.18) we get

$$d\tau = \frac{\sigma_T}{\sqrt{3}} \frac{d\varphi}{\cos\varphi - c\sin\varphi} = \frac{\sigma_T}{\sqrt{3}} \frac{d\Psi}{\cos\Psi} \cos\gamma_0 \tag{3.20}$$

where

$$\Psi = \varphi + \gamma_0, \quad \operatorname{tg}\gamma_0 = c \tag{3.21}$$

Integrating (3.20), we get

$$\tau = \frac{\sigma_T}{\sqrt{3}} \cos\gamma_0 \ln \sqrt{\frac{1+\sin\Psi}{1-\sin\Psi} \frac{1-\sin\gamma_0}{1+\sin\gamma_0}}$$

$$\sigma = \sigma_0 + \sigma_T \sin\gamma_0 \ln \sqrt{\frac{1+\sin\Psi}{1-\sin\Psi} \frac{1-\sin\gamma_0}{1+\sin\gamma_0}} \tag{3.22}$$

Next, using (3.15), we express the residual stresses as

$$s = \frac{2}{3}\left\{\sigma_0 - \sigma_T\cos\varphi + \sigma_T\sin\gamma_0 \ln\sqrt{\frac{1+\sin\Psi}{1-\sin\Psi}\,\frac{1-\sin\gamma_0}{1+\sin\gamma_0}}\right\}$$
$$t = \frac{\sigma_T}{\sqrt{3}}\left\{-\sin\varphi + \cos\gamma_0 \ln\sqrt{\frac{1+\sin\Psi}{1-\sin\Psi}\,\frac{1-\sin\gamma_0}{1+\sin\gamma_0}}\right\} \tag{3.23}$$

Furthermore, expressing $\phi$ from (3.22) in terms of $\tau$ and substituting in (3.23), we arrive at the following final formulas

$$s = \frac{2}{3}\left\{\sigma - \sigma_T\left[\operatorname{th} V_0 \operatorname{th}\tau^* + \frac{1}{\operatorname{ch} V_0 \operatorname{ch}\tau^*}\right]\right\} \tag{3.24}$$
$$t = \tau - \frac{\sigma_T}{\sqrt{3}}\,\frac{\operatorname{sh}\tau^* - \operatorname{sh} V_0}{\operatorname{ch} V_0 \operatorname{ch}\tau^*}$$

where

$$\tau^* = \frac{\sqrt{3}\,\tau}{\sigma_T\cos\gamma_0} + V_0, \quad \operatorname{th} V_0 = \sin\gamma_0 \tag{3.25}$$

The above problems could be also solved for a more general case where the strain hardening is linear but the Bauschinger effect is not ideal. For lack of space, however, we shall not dwell on this any longer. We mention only that the starting relationships in this case have the following form

$$ds = g_0\frac{2/3\,\sigma - s}{T^\circ}\,dT^\circ, \quad dt = g_0\frac{\tau - t}{T^\circ}\,dT^\circ \quad \left(g_0 = \frac{G_2}{G_1}\right) \tag{3.26}$$

where $G_2$ and $G_1$ are the moduli of elastic and plastic strain hardening respectively. After the determination of $s$ and $t$, plastic strains are found from formulas

$$\varepsilon_{11p} = \frac{1}{2G_2}s, \quad \varepsilon_{12}{}^{p} = \frac{1}{2G_2}t \tag{3.27}$$

## 4. Comparison of the theory with experiments

There are very many experimental results on hand on combined loading. The basic conclusions reached from them can be formulated briefly as follows.

1. The experimental results, as a rule, fall between the results computed from the theory of plastic flow and the theory of small plastic deformations.

2. The experimental curves are usually closer to the flow theory curves than to the deformation theory curves. This latter yields especially poor agreement with the experiments if during the process of loading the state of stress is varied rapidly. However, for relatively smooth loading paths the experimental curves are sometimes roughly equally close to the curves obtained from the flow theory as from the deformation theory.

3. There is experimental evidence [8], which contradicts the assump-

tion of the coaxiality of the plastic strain increment tensor with the stress tensor $\sigma_{ij}$.

To compare the proposed theory with experiments we shall use the results obtained in [8, 9]. These results are sufficiently typical.

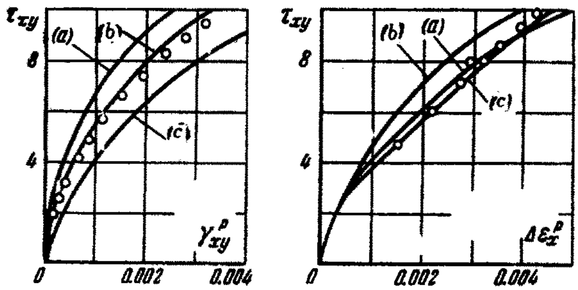


Fig. 6.

Fig. 6 shows strain curves $\epsilon_x{}^p$ and $\epsilon_{xy}{}^p$ which are obtained from a combined loading consisting of two stages: (1), compression to the stress $\sigma_{xx} = -\sigma_0$, and (2), simultaneous action of compression and shear varying in the following way:

$$\frac{d\sigma_{xx}}{d\tau_{xy}} = 0.052$$

Curve $(a)$ corresponds to the flow theory, curve $(b)$ to the deformation theory and curve $(c)$ to the authors' theory in which the Bauschinger effect is considered to be ideal and the strain hardening linear. In the same figure the dots represent experimental data from [9], (p. 510). It can be seen that the theory suggested presently gives results which are closer to the experimental data than either the flow theory or the deformation theory results. Approximately the same picture is obtained from comparison of these three theories with the experimental data given in [9] for other loading paths.

In [8] the loading consisted of the following consecutively alternating stages: (1) extension of a thin-walled tube subjected to a constant torque; (2) extension without torsion, etc. The author of [8] remarks that the second stage of loading was accompanied by a plastic untwisting of specimens, which clearly contradicts the flow theory. This result can be explained only if we drop the assumption that the stress tensor and the plastic strain increment tensor are coaxial. In Fig. 7 the experimental data in [8] of the first two stages of the loading are compared with the flow theory (curve $a$), with the deformation theory (curve $b$) and with the present theory (curve $c$). (Here, as previously, it is assumed that the Bauschinger effect is ideal and the strain hardening is linear.)

It can be seen that the present theory again satisfactorily agrees with the experiments. Moreover, it predicts the untwisting of the specimens, the fact which puzzled the author of [8].

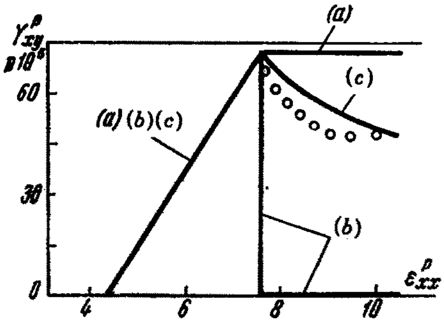


Fig. 7.

To evolve the present theory of plasticity, which accounts for the displacement of the center of the yield surface, it was necessary to introduce concepts of the residual microstresses and the active stresses. In the book by N.N. Davidenkov [5], already quoted, the microstresses are held responsible for the Bauschinger effect, and they are considered to be elastic. In book [10] (pp. 207, 210) reference is made to the microstresses in metals subjected to plastic deformations. Thus, the existence of the microstresses and their basic properties are well known to the metallurgists. However, until now, they were not incorporated into a mathematical description of the theory of plasticity. The present paper closes this gap, and it shows that the inclusion of the microstresses permits an explanation of such facts as the Bauschinger effect, the effect of non-coincidence of principal directions of the increments of plastic strains with the principal directions of the stresses, as was observed in [8]. Finally, it permits us to explain the fact that the experimental curves usually fall between the curves obtained from the flow theory and the deformation theory.

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